Dynamics of Flexible Bodies in Tree Topology—A Computer-Oriented Approach

Ramendra P. Singh* and Richard J. VanderVoort†
Honeywell Incorporated, Clearwater, Florida
and
Peter W. Likins‡
Lehigh University, Bethlehem, Pennsylvania

An approach suited for automatic generation of the equation of motion for large mechanical systems (i.e., large space structures, mechanisms, robots, etc.) is presented. The system topology is restricted to a tree configuration. The tree is defined as an arbitrary set of rigid and flexible bodies connected by hinges characterizing relative translations and rotations of two adjoining bodies. The equations of motion are derived via Kane's method. The resulting equations set is of a minimum dimension. Dynamical equations are imbedded in a computer program called Treetops. Extensive control simulation capability is built into the Treetops program. The simulation is driven by an interactive setup program resulting in an easy-to-use analysis tool.

Introduction

IN the last two decades, considerable efforts have been made in the efficient formulation and solution of the dynamic equations of multibody mechanical systems. Examples of such systems include spacecraft, large space structures, manipulators, mechanisms, and biosystems. This paper deals with an improved approach to the solution fo the dynamics and control problems that arise in each of these areas. This approach accommodates structures with an open-tree topology including flexibility in any of all of the individual bodies, along with large-angle rotations and translations at the hinges. This formulation reduces the problem to minimum dimension set that is very desirable from a computational standpoint.

The emphasis of researchers working with multibody systems has been the expanded generality of mathematical models and the formulation of equations of motion that are amenable to computer solution. Dynamical equations of motion for multibody systems can be derived by Newton-Euler methods or by analytical mechanics methods (e.g., Lagrange, Hamilton canonical, Boltzman-Hamel equations, etc.). The relative advantages and disadvantages of these various approaches depend upon 1) the choice of dependent (kinematic) variables, 2) the geometrical organization and accounting procedure for a unique kinematic description, and 3) the amenability to computer solution. Comparative studies in recent years[1-4] suggest that Kane's method of generalized speed[5] or some related generalization of Lagrange's form of D'Alembert's principle most closely meet these needs.[6] Kane's method combines the computational advantages of both Newton's laws and the Lagrangian formulation—the nonworking constraint forces and torques do not appear and tedious differentiation of scalar energy functions is avoided. The resulting equation set is of minimum dimension. Various methods of analytical dynamics best suited for automatic generation of the equations of motion are reviewed in Ref. 7.

The interest in multibody dynamics modeling has arisen, almost simultaneously, in three distinct fields: spacecraft dynamics, mechanisms, and robotics. In what follows, the extensive literature in multibody dynamics in these three fields will be very briefly reviewed.

In the early years, space vehicles were idealized as rigid bodies or elastic beams for control system design purposes. In response to needs of the aerospace industry in mid-1960s, the equations of motion were published[8,9] for a point-connected set of interconnected rigid bodies in a topological tree. A model combining rigid bodies and elastic appendages was developed and applied extensively[10-12] in the 1970s. Incorporation of body flexibility in the topological tree model was the next logical step, taken in some cases concurrently with the inclusion of relative translation between the bodies.[13] In the most general case considered to date, the prohibition of closed loops in the topology is also relaxed.[14-16]

In parallel with the preceding development, the same problems were being considered for complex machinery with floating links (e.g., reciprocating engines and printing, textile, and agricultural machinery). The "kinetostatic" approach (calculation of the applied forces and reactions associated with an assumed state of the motion with the aid of graphical kinematics) did not seem adequate. Graphically oriented thinking was abandoned to reframe the pertinent problems of kinematics and dynamics in a form better suited to numerical methods of analysis. Since early 1970s, much work has been done in the formulation of dynamical equations of motion for mechanisms with rigid links.[17-21] Based on these formulations, a number of general-purpose computer programs have been developed. But most of these computer programs are limited to analyzing planar machines. The distributed elasticity of the links can be modeled by the introduction of massless elastic springs at selected hinges as in Refs. 17-21. References can be made[22-25] on direct methods for including distributed flexibility.

Researchers in robotics make extensive use of 4×4 transformation matrices of Denavit and Hartenberg.[26] These 4×4 transformations may be interpreted as a combination of rota-

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*Senior Staff Engineer, Space & Strategic Avionics Division. Currently with DYNACS Engineering Company, Clearwater, Fla. Associate Fellow AIAA.

†Senior Staff Engineer, Space & Strategic Avionics Division. Currently with DYNACS Engineering Company, Clearwater, Fla.

‡President. Fellow AIAA.

tion and translation and may be readily used to describe the positions and orientations of coordinate frames in space. Uicker[27] presented a derivation of the exact equations of motion for rigid-link spatial mechanical systems, using 4×4 transformation matrices. These results were specifically adapted for open kinematic chains,[28,29] the most common manipulator configuration. Although these early derivations served as a theoretical framework, the results were too complicated and the computations too costly to be practical. Several simplifications of the basic equations of motion have been made[30,31] to reduce the computation time. From a computational point of view, the most promising methods are the recursive formulations[32,33] presented in the last few years. Some robot formulations include the influence of member flexibility[34,35] and Dubowsky and Gardner[24] have also incorporated a model for joint clearance.

Although all three areas of interest deal with the same class of multibody systems, the equation formulation and corresponding computer programs from one area are in general of little use in the other areas. Particular care was taken in formulating the equations presented in this paper so that the resulting computer program, Treetops, would be useful to all areas of interest that deal with multibody structures in an open-tree topology. Some features of the Treetops program are: 1) any or all bodies can be rigid or deformable; 2) loop closures are permitted if the constraints are kinetic rather than kinematic; 3) the hinges can have zero to six degrees of freedom and hinge rotations can be large; 4) the dimension of the problem equals the number of the degrees of freedom (constraint forces do not appear in the formulation); 5) individual body deformations can be described by any set of modal vectors; 6) an interactive program helps the user set up his problem; 7) extensive control simulation capability (a full set of sensors and actuators, a linear time invariant control law in block diagram format, a nonlinear controller in user supplied subroutine format) is built into the program; and 8) equations can be linearized (via Taylor series expansion) about a nominal state.

The design of Treetops is such that the generic formulation does not compromise the computational efficiency of the program. For example, in the case of a planar mechanism the problem is defined by the user with no out-of-plane degrees of freedom. The computer program, which formulates equations only for the user-defined degrees of freedom, knows only of the planar motion and will perform no computations for out-of-plane motion. Computationally, this is an attractive feature of Treetops.

This paper consists of four sections. The first describes the geometry of open-tree multibody system and the second presents the derivation of motion equations via Kane's method. A guideline for choosing modal displacement vector is given in the third section and an overview of the Treetops program in the fourth. An illustrative example is presented.

System Description

A multibody system in a topological tree configuration is shown in Fig. 1. Body 1 is arbitrarily selected as the reference body. A nonsequential numbering of the bodies has been deliberately indicated in Fig. 1. For convenience, the reference body is assumed to be connected to an imaginary inertially fixed body numbered 0. Development of an accounting procedure for a unique kinematic description requires the knowledge of a direct path from the inertial frame (body 0) to any body of the system. A direct path array can be constructed as follows. To any body j assign a number $c(j)$, which is the body number of the adjacent body leading inboard to body 0 (or the reference body). In Fig. 1, for example, $c(10)=4$ for $j=10$ and $c(1)=0$ for $j=1$. Given j and $c(j)$, one can readily draw Fig. 1. Every j has a unique path set whose elements are j, 0, and all the bodies in the path from j to 0. A set $P^{(j)}$ can be constructed from the knowledge of j and $c(j)$ as a set of all the bodies outboard of j including j and, similarly, $E^{(j)}$ can be defined as another set having all the elements of $P^{(j)}$ excluding j.

A hinge, as shown in Fig. 2, is defined as a pair of two material points, one on each of two adjoining bodies. The discrete degrees of freedom of the jth body in the tree are characterized by the relative translations and rotations of two sets of reference axes located at the hinge points P_j [a point of body $c(j)$] and h_j (a point of body j). Points P_j and h_j and the reference frames at thee two points constitute the jth hinge. A fictitious hinge is assigned to the reference body by assuming P_1 is an inertial point. Thus, the number of hinges equals the number of bodies in the system.

The following symbols are used to describe the configuration:

NB = number of bodies

NT_j = translational degrees of freedom of the jth hinge

NR_j = rotational degrees of freedom of the jth hinge

NM_j = deformational degrees of freedom of the jth body

NS = total number of degrees of freedom of the tree

$$NS = \sum_{j=1}^{NB} (NT_j + NR_j + NM_j)$$

Formulation Methodology

The equations of motion are derived via Kane's method.[5] This approach is expressed in terms of generalized and "quasi-coordinates" (e.g., the scalar components of angular velocity vectors). In a previous paper,[6] this method was employed to derive the equation of motion for structures in a topological chain. In what follows, the approach will be extended to accommodate bodies in a tree configuration.

Figure 3 shows B_k, the kth body in the tree, in a deformed state. An elemental mass dm of B_k is located inertially by

$$\boldsymbol{R}^k = \boldsymbol{R}^{h_k} + \boldsymbol{r}^k + \boldsymbol{u}^k \tag{1}$$

where the bold type represents a vector quantity, $\boldsymbol{r}^k$ is fixed in a reference frame at h_k and $\boldsymbol{u}^k$ represents the elastic deformation. A set of NM_k modal vectors $\boldsymbol{\phi}_\ell^k$, $\ell = 1, \ldots, NM_k$ is chosen to approximately represent the elastic deformation by a finite

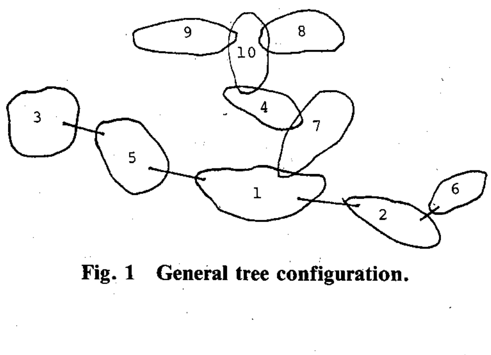

Fig. 1 General tree configuration.

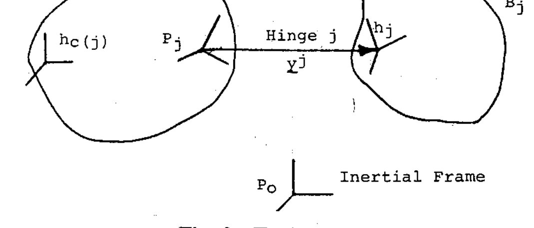

Fig. 2 Typical hinge.

number of kinematic variables η_ℓ^k (modal coordinates) by

$$u^k = \sum_{\ell=1}^{NM_k} \phi_\ell^k(r^k)\eta_\ell^k \tag{2}$$

Time differentiation in inertial space provides

$$\dot{R}^k = \dot{R}^{h_k} + \omega^k \times (r^k + u^k) + \overset{\circ}{u}^k \tag{3}$$

where ($^\circ$) implies time differentiation in the local reference frame (here b_k), and w^k is the inertial angular velocity vector of the frame b_k. Since the modal vectors in Eq. (2) are fixed in the frame b_k, the relative deformation velocity is given by

$$\overset{\circ}{u}^k = \sum_{\ell=1}^{NM_k} \phi_\ell^k(r^k)\dot{\eta}_\ell^k \tag{4}$$

Let a set of generalized speeds (time derivatives of coordinates characterizing the configuration) $w_1,\ldots, w_{NS}$ represent the degrees of freedom of the system. Then $\dot{R}^k$ can always be written as

$$\dot{R}^k = \sum_{p=1}^{NS} V_p^k w_p + V_t^k \tag{5}$$

where V_p^k and V_t^k are functions of NS generalized coordinates and time.

Similarly, express $\dot{R}^{h_k}$, ω^k, and $\overset{\circ}{u}^k$ as

$$\dot{R}^{h_k} = \sum_{p=1}^{NS} V_p^{h_k} w_p + V_t^{h_k} \tag{6}$$

$$\omega^k = \sum_{p=1}^{NS} \omega_p^k w_p + \omega_t^k \tag{7}$$

$$\overset{\circ}{u}^k = \sum_{p=1}^{NS} V_p^{\dot{\eta}_k} w_p \tag{8}$$

Substituting the above in Eq. (3), one obtains

$$\dot{R}^k = \sum_{p=1}^{NS} V_p^{h_k} w_p + V_t^{h_k} + \left(\sum_{p=1}^{NS} \omega_p^k w_p + \omega_t^k\right) \times \left(r^k + \sum_{\ell=1}^{NM_k} \phi_\ell^k(r^k)\eta_\ell^k\right) + \sum_{p=1}^{NS} V_p^{\dot{\eta}_k} w_p \tag{9}$$

Note that $V_p^{\dot{\eta}_k}$ is zero unless the generalized speed w_p corresponds to one of the modal velocities $\dot{\eta}_\ell^k$ in body B_k.

Comparison of Eqs. (5) and (9) provides

$$V_p^k = V_p^{h_k} + \omega_p^k \left[r^k + \sum_{\ell=1}^{NM_k} \phi_\ell^k(r^k)\eta_\ell^k\right] + V_p^{\dot{\eta}_k} \tag{10a}$$

and

$$V_t^k = V_t^{h_k} + \omega_t^k \left[r^k + \sum_{\ell=1}^{NM_k} \phi_\ell(r^k)\eta_\ell^k\right] \tag{10b}$$

The equations of motion for the tree structure can be written starting with Newton's law for the differential mass element dm, which is located by the vector R^k

$$\mathrm{d}f - \ddot{R}^k \mathrm{d}m = 0 \tag{11}$$

where df is the force on the differential element and $\ddot{R}^k$ the inertial acceleration of dm.

The equations of motion for the tree structure can be written as

$$\sum_{k=1}^{NB} \int_{B_k} V_p^k \cdot (\mathrm{d}f - \ddot{R}^k \mathrm{d}m) = 0, \quad p = 1,\ldots,NS \tag{12}$$

or,

$$\sum_{k=1}^{NB} \int_{B_k} V_p^k \cdot \mathrm{d}f - \sum_{k=1}^{NB} \int_{B_k} V_p^k \cdot \ddot{R}^k \mathrm{d}m = 0, \quad p = 1,\ldots,NS \tag{13}$$

Equation (12) can be expressed as a system of NS equations given by

$$f_p + f_p^* = 0, \quad p = 1,\ldots,NS$$

where f_p and f_p^* are the generalized active and the generalized inertia forces, respectively.

The expression for the pth generalized active force can be derived by substituting Eq. (10a) into Eq. (12),

$$f_p = \sum_{k=1}^{NB} \int_{B_k} V_p^k \cdot \mathrm{d}f = \sum_{k=1}^{NB} \left(M^{h_k} \cdot \omega_p^k + F^k \cdot V_p^{h_k} + \int_{B_k} V_p^{\dot{\eta}_k} \cdot \mathrm{d}f\right) \tag{14}$$

where M^{h_k} is the moment on B_k with respect to the hinge point h_k of the working forces (eliminating nonworking constraint forces) and F^k is the force on B_k obtained by summing the working forces (eliminating nonworking constraint forces). The last term in Eq. (14) accounts for the contributions of internal elastic forces in body B_k and any external forces acting on B_k. For each p corresponding to one of the modal velocities in B_k, the integral represents the negative of a row of the generalized stiffness (and damping) matrix times the column of modal coordinates (and modal velocities).

Similarly, the expression for the pth generalized inertia force is

$$-f_p^* = \sum_{k=1}^{NB} \int_{B_k} V_k^p \cdot \ddot{R}^k \mathrm{d}m$$

$$= \sum_{k=1}^{NB} \Big[m_k(\ddot{R}^{h_k} + \ddot{\ell}^k) \cdot V_p^h k + (\dot{H}^{h_k} + m_k \ell^k \times \ddot{R}^{h_k}) \cdot \omega_p^k + \ddot{R}^{h_k} \cdot \int V_p^{\dot{\eta}_k} \mathrm{d}m + \dot{\omega}^k \cdot \int_{B_k} (r^k + u^k) \times V_p^{\dot{\eta}_k} \mathrm{d}m + 2\omega^k \cdot \int_{B_k} \overset{\circ}{u}^k \times V_p^{\dot{\eta}_k} \mathrm{d}m + \int_{B_k} \overset{\circ\circ}{u}^k \cdot V_p^{\dot{\eta}_k} \mathrm{d}m - \omega^k \cdot \underline{D}_p^k \cdot \omega^k \Big] \tag{15}$$

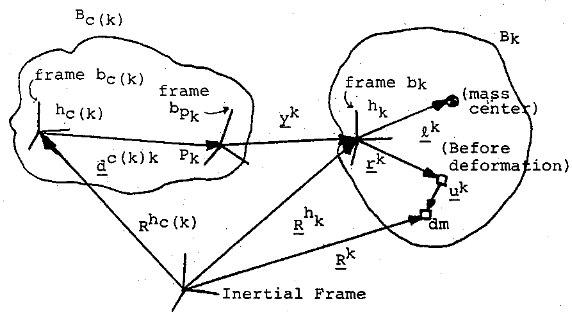

Fig. 3 Geometry of deformable body j.

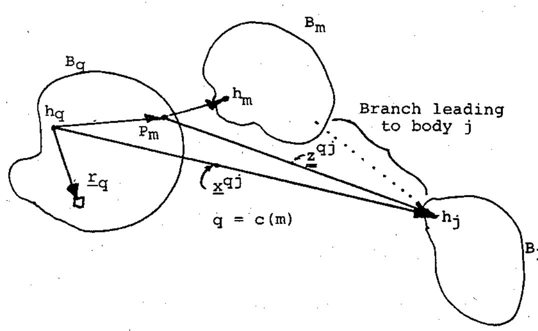

Fig. 4 Vectors defining coefficients of generalized speed.

where m_k is the mass of B_k, ℓ^k locates the mass center of the deformed body B_k with respect to h_k, $(\dot{\ })$ and $(\ddot{\ })$ symbolize the first and second derivative in the inertial frame, respectively, $(\mathring{\mathring{\ }})$ indicates the second derivative in the frame b_k (Fig. 3), u^k is defined by Eq. (2), and $\dot{H}^hk$ and $\underline{D}_p^k$ (underbar implies a dyadic) are defined below.

H^{h_k} is the angular momentum of B_k with respect to h_k

$$H^{h_k}=\int_{B_k}(r^k+u^k)\times(\dot{r}^k+\dot{u}^k)\mathrm{d}m \tag{16}$$

By forming the time derivatives of H^{h_k} in the inertial frame, we find

$$\begin{aligned}\dot{H}^{h_k}=&\underline{I}^{h_k}\cdot\dot{\omega}^k+\omega^k\times\underline{I}^{h_k}\cdot\omega^k\\ &+\sum_{\ell=1}^{NM_k}\{h_\ell^k\ddot{\eta}_\ell^k+[(\underline{N}_\ell^k+\underline{M}_\ell^k)\cdot\dot{\omega}^k]\eta_\ell^k\\ &+2\underline{N}_\ell^k\cdot\omega^k\eta_\ell^k+\omega^k\times[\underline{N}_\ell^k+\underline{M}_\ell^k)\cdot\omega^k\eta_\ell^k\}\\ &+\int_{B_k}\{u^k\times\mathring{\mathring{u}}^k+u^k\times(\dot{\omega}^k\cdot u^k)\\ &+2u^k\times(\omega^k\times\mathring{u}^k)+u^k\times[\omega^k\cdot(\omega^k\times u^k)]\}\mathrm{d}m\end{aligned} \tag{17}$$

where $\underline{I}^{h_k}$ is the inertia dyadic of B_k (*undeformed*) referred to h_k and h_ℓ^k and the dyadic quantities $\underline{N}_\ell^k$ and $\underline{M}_\ell^k$ are defined as

$$h_\ell^k=\int_{B_k}r^k\times\phi_\ell^k(r^k)\mathrm{d}m \tag{18}$$

$$\underline{N}_\ell^k=\int_{B_k}[(r^k\cdot\phi_\ell^k(4r^k)\underline{U}-r^k\phi_\ell^k(r^k)]\mathrm{d}m \tag{19}$$

$$\underline{M}_\ell^k=\int_{B_k}[(\phi_\ell^k(r^k)\cdot r^k)\underline{U}-\phi_\ell^k(r^k)r^k]\mathrm{d}m \tag{20}$$

where $\underline{U}$ is the unit dyadic.

D_p^k, appearing in Eq. (15), is given as

$$\begin{aligned}D_p^k=\int_{B_k}\Big\{&\Big[r^k+\sum_{\ell=1}^{NM_k}\phi_\ell^k(r^k)\eta_\ell^k\Big]\cdot V_p^{\eta_k}\underline{U}\\ &-\Big[r^k+\sum_{\ell=1}^{NM_k}\phi_\ell^k(r^k)\eta_\ell^k\Big]\underline{V}_p^{\eta_k}\Big\}\mathrm{d}m\end{aligned} \tag{21}$$

The mass center location of B_k in a deformed state can be expressed by

$$\ell^k=\frac{1}{m_k}\int_{B_k}\Big[r^k+\sum_{\ell=1}^{NB_k}\phi_\ell^k(r^k)\eta_\ell^k\Big]\mathrm{d}m \tag{22}$$

Substitution of Eqs. (14) and (15) into Eq. (13) provides the basic equations of motion for a system of *NB* flexible bodies. From Fig. 3,

$$R^{h_k}=R^{h_c}(k)+d^{c(k)k}+y^k \tag{23}$$

where

$$d^{c(k)k}=D^{c(k)k}+\sum_{\ell=1}^{NM_c(k)}\phi_\ell^{c(k)}(p_k)\eta_\ell^{c(k)}$$

and where $D^{c(k)k}$ is the vector from $h_{c(k)}$ to p_k prior to the deformation of $B_{c(k)}$ and $\phi_\ell^{c(k)}(p_k)$ are the modal displacements (fixed in $B_{c(k)}$ frame) at p_k.

The time derivative of R^{h_k} in the inertial frame is given by

$$\begin{aligned}\dot{R}^{h_k}=&\dot{R}^{h_{c(k)}}+\sum_{\ell=1}^{NM_{c(k)}}\phi_\ell^{c(k)}(p_k)\dot{\eta}_\ell^{c(k)}\\ &+\omega^{c(k)}\times d^{c(k)k}+\mathring{y}^k+\Big[\omega^{c(k)}+\sum_{\ell=1}^{NM_{c(k)}}{}_\ell^{\prime c(k)}(p_k)\dot{\eta}_\ell^{c(k)}\Big]\times y^k\end{aligned} \tag{24}$$

where $\mathring{y}^k$ is the time derivative of y^k in the frame b_{p_k} located at p_k (Fig. 3), $w^{c(k)}$ the angular velocity of the frame $b_{c(k)}$ and $\phi_\ell^{c(k)}(p_k)$ the modal rotations at p_k.

The angular velocity of the frame b_k can be expressed as

$$\omega^k=\omega^{c(k)}+\sum_{\ell=1}^{NM_{c(k)}}\phi_\ell^{\prime c(k)}(p_k)\dot{\eta}_\ell^{c(k)}+\omega^{kc(k)} \tag{25}$$

where $\omega^{kc(k)}$ is the rotational velocity vector of b_k relative to b_{pk}.

Notice that $\dot{R}^{h_k}$ and ω^k (and similarly $\ddot{R}^{h_k}$ and $\dot{\omega}^k$) have been written in a recursive form. Thus, one can express $\dot{R}^{h_k}$, ω^k, $\ddot{R}^{h_k}$, and $\dot{\omega}^k$ for $k=1,\ldots,NB$ simply by knowing k and $c(k)$.

The following choice of the generalized speeds is made:

1) $\dot{\theta}_m^k$ for the kth hinge relative rotational velocity

$$\omega^{kc(k)}=\sum_{m=1}^{NR_k}\dot{\theta}_m^k\ell_m^k$$

where ℓ_m^k is a unit vector along the mth gimbal (rotational) axis of the kth hinge.

2) $\mathring{y}_n^k$ for the kth hinge relative translational velocity

$$\mathring{y}^k=\sum_{n=1}^{NT_k}\mathring{y}_n^k g_n^k$$

where g_n^k is a unit vector, fixed in b_{p_k}, along the translational degree of freedom of the kth hinge.

3) $\dot{\eta}_1^k,\ldots,\dot{\eta}_{NM_k}^k$ for the deformation rates of the kth body.

The coefficients of generalized speeds, $V_p^{h_k}$, ω_p^k, and $V_p\dot{\eta}k$, can be obtained from Eqs. (24) and (25).

Let p in Eqs. (14) and (15) correspond to the mth rotational axis of the qth hinge. Then,

$$\begin{aligned}V_p^{h_j}&=\ell_m^q\cdot x^{qj}\quad\text{for } j\in P_{(q)}\\ &=0\quad\text{otherwise}\\ \omega_p^j&=\ell_m^q\quad\text{for } j\in E_{(q)}\\ &=0\quad\text{otherwise}\\ V_p^{\eta j}&=0\end{aligned} \tag{26}$$

x^{qj} is shown in Fig. 4.

Let p in Eqs. (14) and (15) represent the translational degree of freedom along the nth axis of the qth hinge (along q_n^q),

$$V_p^{hj} = g_n^q \quad \text{for } j \in E(q)$$
$$= 0 \quad \text{otherwise}$$
$$\omega_p^j = 0$$
$$V_p^{\dot{\eta} j} = 0 \tag{27}$$

Let p in Eqs. (14) and (15) correspond to the ℓ th modal degree of freedom of the qth body,

$$V_p^{hj} = \phi_\ell^q(Pm) + \phi'^q_\ell(Pm) \cdot Z^{qj} \quad \text{for } j \in P_{(q)}$$
$$= 0 \text{ otherwise}$$
$$\omega_p^j = \phi'^q_\ell(Pm) \quad \text{for } j \in P_{(q)}$$
$$= 0 \quad \text{otherwise}$$
$$V_p^{\dot{\eta} j} = \phi_\ell^q(r^q) \quad \text{for } j = q$$
$$= 0 \text{ for } j \neq q \tag{28}$$

where $\phi_\ell^q(p_m)$ and $\dot{\phi}_\ell^q(p_m)$ are the modal displacements and the modal rotations at p_m (see Fig. 4), respectively. $\phi_\ell^q(r^q)$ represents the modal displacement at a generic point of the qth body and z^{qj} is shown in Fig. 4.

Equations (26-28) are the coefficients of generalized speeds identified by p, the kinematical coordinate associated with the tree configuration. When substituted into Eqs. (14) and (15), they yield a complete set of equations of motion for the topological tree system.

Choice of Mode Shapes

Any set of modal displacement vectors $\phi_\ell^k(r^k)$ (ℓth mode of kth body) can be chosen that satisfy the following requirements:

$$\phi_\ell^k(h_k) = \acute{\phi}_\ell^k(h_k) = 0$$

where point h_h is the reference point of the kth body as shown in Fig. 3.

For spacecraft application (unrestrained motion of the multibody system), the mode shapes for the reference body (say B_1) might be chosen as the "free-free" modes with h_1 being the mass center of B_1 and, for bodies beyond B_1, one might choose the "fixed-free" modes with translation and rotation at h_k constrained. Alternatively, one might use the "augmented body" concept as defined in Refs. 8 and 9 and the "free-free" modes of the remaining augmented bodies. In an open topology, an "augmented body" is the body itself augmented at the points of connection to other bodies by particles, each having the mass of the set of bodies attached through that connection point. Modal coordinates of augmented bodies may be preferable to those for unaugmented bodies because the former reflect at least in part the influence of the inertial loading of the attached bodies. Care must be taken to ensure that the augmented mass elements are not doubly accounted for.

Computer Program (Treetops)

The simulation consists of three major parts: a tree topology of flexible structures, a controller, and a set of sensors and actuators.

Tree Topology

Based on Eqs. (14-15), the simulation numerically forms the first-order differential equations

$$\dot{x}_p = M^{-1} y_p \tag{29}$$
$$\dot{z}_p = x_p \tag{30}$$

where x_p is the system state vector (dimensioned NS) formed of all the generalized speeds, $\dot{x}_p$ and z_p are the first derivative (second derivative of kinematic variables) and integral of the state vector, respectively, and m is the system "inertia" matrix.

Controller

The emphasis of the simulation is on high-fidelity modeling of multibody structures, but the intent of the effort is to build a control system design tool. To this end, a set of sensors and actuators is built into the simulation along with a controller specified in a linear, block diagram format or a user-supplied subroutine.

Linear, Block-Diagram Controller Format

This format is intended to provide a quick, first-cut analytical capability. The controller is a multi-input, multioutput system composed of transfer functions and summing junctions, with gains included in the interconnections between elements. The block diagram is reduced within the simulation to a set of matrix quadruples A, B, C, and D corresponding to the vector form of the control equations

$$\dot{x} = Ax + Bu \tag{31}$$
$$r = Cx + Du \tag{32}$$

where x, u, and r are the controller state, input, and output vectors, respectively, and A, B, C, D the constant coefficient matries.

When the linear controller is continuous, Eq. (31) is integrated simultaneously with the structure equations (29) and (30). A discrete linear controller may also be defined in which case the discrete equations corresponding to Eqs. (31) and (32) are updated only at the sample times.

User Supplied Control Subroutine

For more advanced control studies where the user wishes to try optimal, nonlinear, or time-varying controllers, he may supply his own control subroutine. This is also a multi-input, multioutput system and it offers unrestricted control capability, but the burden of software development is on the user. It should be noted that the continuous linear controller, the discrete controller, and the user-supplied surbroutine may all be used simultaneously and interconnected as desired.

Sensors and Actuators

A set of seven sensors has been built into the simulation (rate gyro, accelerometer, position and velocity sensors, a tachometer). The user needs to define only the type of sensor, its mounting point node, and input axis orientation. Likewise, a set of four actuators has been defined (reaction jet, hydraulic cylinder, momentum wheel and torque motor). The user can generate disturbances or control forces (torques) on the structure by simply specifying heat actuator type, mounting point node, output axis orientation, and actuator command input. The user need not be concerned with how the actuator output forces enter into the M^{hj}, F^j, and df terms in Eq. (14).

The simulation has an option for computing a linear structure model for preliminary controller design using off-line linear analysis tools. In this sense, the structure includes the sensors and actuators because control analysis requires end-to-end transfer functions from actuator inputs to sensor outputs.

In control terminology, the "plant" consists of the structure, sensors, and actuators and is described by the nonlinear differential equations,

$$\dot{x}_p = f_x(x_p, z_p, u_p) = M^{-1} y_p \tag{33}$$

$$\dot{z}_p = x_p \tag{34}$$

$$r_p = f_r(x_p, z_p, y_p) \tag{35}$$

where u_p and r_p are the plant input and output vectors and f_x and f_r are nonlinear vector functions.

Linearization is accomplished by expanding Eqs. (33) and (35) [Eq. (34) is already linear] about the nominal operating point (x_p, z_p, u_p) and retaining only the first-order terms. Thus,

$$\delta\dot{x}_p = A_P \delta x_p + B_P \delta z_p + C_P \delta u_P \tag{36}$$

$$\delta r_P = D_P \delta x_P + E_P \delta z_P + F_P \delta u_P \tag{37}$$

$$A_p = \frac{\partial f_x}{\partial x_p}, \quad B_p = \frac{\partial fx}{\partial z_p}, \quad C_p = \frac{\partial fx}{\partial u_p} \tag{38}$$

$$D_p = \frac{\partial fr}{\partial x_p}, \quad E_p = \frac{\partial fr}{\partial z_p}, \quad F_p = \frac{\partial f_r}{\partial u_p} \tag{39}$$

where A_p, B_p, C_p, D_p, E_p, and F_p are constant coefficient matrices obtained by numerically computing each of the partial derivatives at the nominal operating point. A typical matrix element is found by computing f_{xi} at three values of x_{pj} (x_{pj0}, $x_{pjo} \pm h$) with all other x_p elements at their nominal values. A second-order Lagrance interpolating polynomial is fitted to these three points and the partial derivative is the derivative of the interpolating function evaluated at x_{pj0}.

Treetops is operational at Honeywell and also at the NASA Marshall Space Flight Center where it is now being used to study the control problems relating to the instrument pointing system (IPS), pinhole-occulter experiment, and other spacecraft projects.

Conclusion

An efficient computer-oriented method for formulating equations of motion for large mechanical systems in a topological tree has been presented. It is shown that these equations can be obtained by recursive relations. Nonworking constraint forces do not appear in the formulation. The resulting equation set is of minimum dimension. The emphasis of the simulation presented has been on the high-fidelity modeling of the structure with the intent that the simulation be used for control system studies. The dynamic equations have been formulated and programmed in a general sense. Furthermore, the simulation has an easy-to-use, interactive setup program that creates all the necessary data files. Also included in the simulation are easy-to-use sensor and actuator models.

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References

[1]Kane, T.R. and Levinson, D.A., "Formulation of Equations of Motion for Complex Spacecraft," *Journal of Guidance and Control,* Vol. 3, March-April 1980, pp. 99-112.

[2]Likins, P.W., "Analytical Dynamics and Non-Rigid Spacecraft Simulation," Jet Propulsion Laboratory, California Institute of Technology, Pasadena, Rept. TR 32-1593, July 1974.

[3]Likins, P.W., "Point-Connected Rigid Bodies in a Topological Tree," *Celestial Mechanics,* Vol. 11, 1975, pp. 301-317.

[4]Levinson, D.A., "Equations of Motion for Multiple-Rigid Body Systems via Symbolic Manipulation," *Journal of Spacecraft and Rockets,* Vol. 14, Aug. 1977, pp. 479-487.

[5]Kane, T.R. and Wang, C.F., "On the Derivation of Equations of Motion," *SIAM Journal on Applied Mathematics,* Vol. 13, No. 2, June 1965.

[6]Singh, R.P. and Likins, P.W., "Manipulator Interactive Design with Interconnected Flexible Elements," *Proceedings 1983 Automatic Control Conference,* San Francisco, June 1983.

[7]Paul, B., "Analytical Dynamics of Mechanisms—A Computer Oriented Overview," *Mechanism and Machine Theory,* Vol. 10, 1975, pp. 481-507.

[8]Hooker, W.W. and Margulies, G., "Dynamical Attitude Equations for an *n*-Body Satellite," *Journal of the Astronautical Sciences,* Vol. 12, 1965, pp. 123-128.

[9]Roberson, R.E. and Wittenburg, J., "A Dynamical Formalism for an Arbitrary Number of Interconnected Rigid Bodies, with Reference to the Problem of Satellite Attitude Control," *Proceedings Third Congress IFAC,* Butterworth, London, 1966, pp. 46D.1-46D.8.

[10]Likins, P.W., "Dynamics and Control of Flexible Space Vehicles," Jet Propulsion Laboratory, California Institute of Technology, Pasadena, Calif. Rept. TR 32-1329, 1970.

[11]Likins, P.W., "Dynamic Analysis of a System of Hinge-Connected Rigid Bodies with Non-Rigid Appendages," *International Journal of Solids and Structures,* Vol. 9, 1973, pp. 1473-1487.

[12]Ho, J.Y.L., "Direct Path Method for Flexible Multibody Spacecraft Dynamics," *Journal of Spacecraft and Rockets,* Vol. 14, Feb. 1977, pp. 102-110.

[13]Roberson, R.E., "A Form of the Translational Dynamical Equations for Relative Motion in Systems of Many Non-Rigid Bodies," *Acta Mechanica,* Vol. 14, 1972, pp. 297-308.

[14]Bodley, C., Devers, A., Park, A., and Frisch, H., "A Digital Computer Program for Dynamic Interaction and Simulation of Controls and Structures (DISCOS)," NASA Tech. Paper 1219, May 1978.

[15]Boland P., Smain, J., and Willems, P., "Stability Analysis of Interconnected Deformable Bodies in a Closed-Loop Configuration," *AIAA Journal,* Vol. 13, 1975, pp. 864-867.

[16]Keat, J.E., "Dynamic Equations of Multibody System with Application to Space Structure Deployment," The Charles Stark Draper Lab., Inc., Cambridge, Mass., Rept. CSDL-T-822, May 1983.

[17]Chace, M.A. and Bayazitoglu, Y.O., "Development and Application of a Generalized D'Alembert Force for Multifreedom Mechanical Systems," *Journal of Engineering for Industry,* Vol. 93, Feb. 1971, pp. 317-327.

[18]Vance, J.M. and Sitchin, A., "Derivation of First-Order Difference Equation for Dynamical Systems by Direct Application of Hamilton's Principle," *Journal of Applied Mechanics,* Vol. 37, 1970.

[19]Vukobratovic, M., "Computer Method for Dynamic Model Construction of Active Articulated Mechanism Using Kinetostatic Approach," *Mechanism and Machine Theory,* Vol. 13, 1978, pp. 19-39.

[20]Haug, E.J., Wehage, R.A., and Barman, N.C., "Dynamic Analysis and Design of Constrained Mechanical Systems," *Journal of Mechanical Design,* Vol. 103, 1981, pp. 560-570.

[21]Nikravesh, P.E. and Chung, I.S., "Application of Euler Parameters to the Dynamic Analysis of Three-Dimensional Constrained Mechanical Systems," *Journal of Mechanical Design,* Vol. 104, 1982, pp. 785-791.

[22]Winfrey, R.C., "Elastic Link Mechanism Dynamics," *Journal of Engineering for Industry,* Vol. 93, 1971, pp. 268-272.

[23]Erdman, A.G., Sandor, G.N., and Oakberg, R.G., "A General Method for Kineto-Elastodynamics Analysis and Synthesis of Mechanisms," *Journal of Engineering for Industry,* Vol. 94, 1972, pp. 1193-1205.

[24]Dubowsky, S. and Gardner, T.N., "Design and Analysis of Multi-Link Flexible Mechanisms with Multiple Clearance Connections," *Journal of Engineering for Industry,* Vol. 99, 1977, pp. 88-96.

[25]Sunada, W. and Dubowsky, S., "The Application of Finite Element Methods to the Dynamic Analysis of Spatial and Coplanar Linkage Systems," *Journal of Mechanical Design,* Vol. 103, 1981, pp. 643-651.

[26]Denavit, J. and Hartenberg, R.S., "A Kinematic Notation for Lower-Pair Mechanisms Based on Matrices," *Journal of Applied Mechanics,* Vol. 22, June 1955, pp. 215-221.

[27]Uicker, J.J. Jr., "On the Dynamic Analysis of Spatial Linkages Using 4×4 Matrices," Ph.D. Dissertation, Northwestern University, Evanston, Ill., 1965.

[28]Kahn, M.E., Roth, B., "The Near-Minimum Time Control of Open-Loop Articulated Kinematic Chains," Stanford University, Stanford, Calif., Artificial Intelligence Memo 106, Dec. 1969.

[29]Yang, A.T., "Inertia Force Analysis of Spatial Mechanisms," *Journal of Engineering for Industry,* Vol. 93, Feb. 1971.

[30]Bejczy, A.K., "Robot Arm Dynamics and Control," NASA TM 33-669, Feb. 1974.

[31]Raibert, M.H., "Mechanical Arm Control Using a State-Space Memory," Society of Manufacturing Engineers Technical Paper MS-77-742. 1977.

[32]Paul, R.P., "Advanced Industrial Robot Control Systems," Purdue University, West Lafayette, Ind., Rept. TR-EE-78-25, 1978.

[33]Hollerbach, J.M., "A Recursive Lagrangian Formulation of Manipulator Dynamics and a Comparative Study of Dynamics Formulation Complexity," *IEEE Transactions on Systems, Man and Cybernetics,* Vol. SMC-10, No. 11, 1980, pp. 730-736.

[34]Book, W.J., "Modeling, Design and Control of Flexible Manipulator Arms," Ph.D Thesis, Massachusetts Institute of Technology, Cambridge, April 1974.

[35]Book, W.J., "Recursive Lagrangian Dynamics of Flexible Manipulator Arms via Transformation Matrices," *IFAC Symposium on Computer Aided Design of Multivariable Tehcnological Systems,* Purdue University, West Lafayette, Ind., Sept. 1982.